

# TESTS ON VARIANCES

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# OUTLINE

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- **Introduction to ANOVA**
- **One-way ANOVA**
- **Single Sample**
- **Two Samples**
- **Three or more samples**

# INTRODUCTION TO ANOVA

- Type I Error and Multiple Hypothesis Tests
- The Logic of ANOVA
- ANOVA vocabulary, notation, and formulas
- The F table
- Comparing pairs of means
- Relationship between ANOVA and t-tests

# ANOVA

- When?
  - The independent variable in a study is categorical (e.g. treatment groups) and the dependent variable is continuous.
- A t-test is used when there are only two groups to compare.
- ANOVA is used when there are two or more groups

# ANOVA - TERMINOLOGY

- In analysis of variance, an independent variable is called a **FACTOR**. (e.g. treatment)
- The groups that make up the independent variable are called **LEVELS** of that factor. (e.g. low dose, high dose, control)
- A research study that involves only one factor is called a **SINGLE-FACTOR** design.
- A research study that involves more than one factor is called a **FACTORIAL** design (e.g. treatment group and gender)

# ANOVA - EXAMPLE

- A researcher want to test tooth sensibility to different temperatures. Three temperatures were investigated:  $0^{\circ}\text{C}$ ,  $5^{\circ}\text{C}$ , and  $10^{\circ}\text{C}$  on a sample of 5 children. The sensibility was stored on a scale from 0 to 6.
- Factor = temperature
- Levels: 0, 5, 10

$0^{\circ}\text{C}$	$5^{\circ}\text{C}$	$10^{\circ}\text{C}$
4	1	0
3	2	1
6	2	3
3	0	1
4	0	0
$m_1 = 4$	$m_2 = 1$	$m_3 = 1$

# TYPE I ERROR AND MULTIPLE TESTS

- Compare 0°C sample and 5°C sample
- Compare 0°C sample and 10°C sample
- Compare 5°C sample and 10°C sample
  
- With an  $\alpha = 0.05$ , there is a 5 % risk of a Type I error.
  - Thus, for every 20 hypothesis tests, you expect to make one Type I error.
  - **The more tests you do, the more risk there is of a Type I error**

# TYPE I ERROR AND MULTIPLE TESTS

- Test-wise alpha level: The alpha level you select for each individual hypothesis test
- Experiment-wise alpha level: The total probability of a Type I error accumulated from all of the separate tests in the experiment.
- Three t-tests, each at  $\alpha = 0.05 \rightarrow$  an experiment-wise alpha of 0.14



# ANOVA

- ANOVA uses one test with one alpha level to evaluate all the mean differences, thereby avoiding the problem of an inflated experiment-wise alpha level
- $H_0: \mu_1 = \mu_2 = \mu_3$
- $H_a$ : at least one population mean is different

# ANOVA

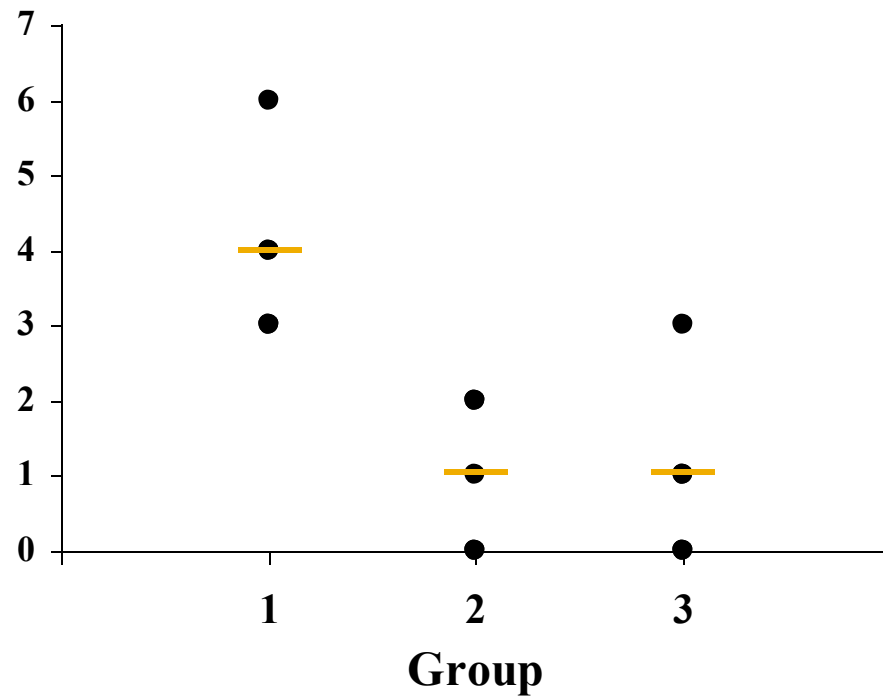
- There is some variability among the means of tooth sensibility
- Is it more than we might expect from chance alone?

0°C	5°C	10°C
4	1	0
3	2	1
6	2	3
3	0	1
4	0	0
$m_1 = 4$	$m_2 = 1$	$m_3 = 1$

# ANOVA

- T-statistic:
  - $t = (\text{obtained difference between sample means}) / (\text{difference expected by chance})$ 
    - obtained difference between sample means - Random variation + Effect
    - difference expected by chance - Random variation
- F-statistic:
  - $F = (\text{obtained variance between sample means}) / (\text{variance expected by chance})$

# ANOVA



# ANOVA

- F-statistic:
  - $F = (\text{obtained variance between sample means}) / (\text{variance expected by chance})$
  - $F = (\text{Between-Group Variance} - s_b^2) / (\text{Within-Group Variance} - s_w^2)$ 
    - Within-Groups Variance: A summary of how much each data-point varies from its own mean
      - $s_w^2 = (SS_{\text{within}}) / df_{\text{within}}$

# ANOVA

- There is some variability among the means of tooth sensibility?
- Is it more than we might expect from chance alone?

0°C	5°C	10°C
4	1	0
3	2	1
6	2	3
3	0	1
4	0	0
$m_1 = 4$	$m_2 = 1$	$m_3 = 1$
$SS_1 = \sum_{i=1}^5 (X_i - m_1)^2 = 6$	$SS_2 = 4$	$SS_3 = 6$

# ANOVA

- Within-treatment variability:
  - A summary of how much each data-point varies from its own treatment mean

$$s_w^2 = \frac{SS_1 + SS_2 + SS_3}{df_1 + df_2 + df_3} = \frac{SS_{\text{within}}}{df_{\text{within}}}$$

$$s_w^2 = \frac{6 + 4 + 6}{4 + 4 + 4} = 1.33$$

0°C	5°C	10°C
4	1	0
3	2	1
6	2	3
3	0	1
4	0	0
$m_1 = 4$	$m_2 = 1$	$m_3 = 1$
$SS_1 = 6$	$SS_2 = 4$	$SS_3 = 6$

# ANOVA

- Between-group variance:
  - A summary of how much each treatment mean varies from the grand mean
  - $SS_{\text{between}}$  = the deviation between each treatment mean and the “grand mean”.
  - Square those deviations
  - And weight them by the sample size of each treatment group
  - Now add them up to get  $SS_{\text{between}}$

0°C	5°C	10°C	
4	1	0	
3	2	1	
6	2	3	
3	0	1	
4	0	0	
$m_1 = 4$	$m_2 = 1$	$m_3 = 1$	$m = 1.6$
$m_1 - m = 2.4$	$m_2 - m = -0.6$	$m_3 - m = -0.6$	
$(m_1 - m)^2 = 5.76$	$(m_2 - m)^2 = 0.36$	$(m_3 - m)^2 = 0.36$	
$5 * 5.76 = 28.8$	$5 * 0.36 = 1.8$	$5 * 0.36 = 1.8$	$SS_{\text{between}} = 32.4$

$$S_b^2 = \frac{SS_{\text{between}}}{df_{\text{between}}} = \frac{\sum n_i (m_i - m)^2}{df_{\text{between}}}$$



# ANOVA

- Between-group variance:  $s_b^2 = \frac{SS_{\text{between}}}{df_{\text{between}}} = \frac{\sum n_i (m_i - m)^2}{df_{\text{between}}} = \frac{32.4}{3-1} = 16.2$

0°C	5°C	10°C	
4	1	0	
3	2	1	
6	2	3	
3	0	1	
4	0	0	
$m_1 = 4$	$m_2 = 1$	$m_3 = 1$	$m = 1.6$
$m_1 - m = 2.4$	$m_2 - m = -0.6$	$m_3 - m = -0.6$	
$(m_1 - m)^2 = 5.76$	$(m_2 - m)^2 = 0.36$	$(m_3 - m)^2 = 0.36$	
$5 * 5.76 = 28.8$	$5 * 0.36 = 1.8$	$5 * 0.36 = 1.8$	$SS_{\text{between}} = 32.4$

# ANOVA

- So what is our test statistic?
- F statistic
- $F = s_b^2 / s_w^2$ 
  - $s_b^2 = 16.2$
  - $s_w^2 = 1.33$
- $F = 16.2 / 1.33 = 12.18$
- What are its **degrees of freedom**?
  - There are two!
  - One for the numerator ( $df_{\text{between}}$ )
  - One for the denominator ( $df_{\text{within}}$ )

# ANOVA

- F statistic
  - Obtained variance between sample means
  - Obtained variance expected by chance
  - $F = \text{between treatment variance } (s_b^2) / \text{within treatment variance } (s_w^2)$ 
    - $s_b^2 = SS_{\text{between}} / df_{\text{between}}$
    - $s_w^2 = SS_{\text{within}} / df_{\text{within}}$
  - → there are 2 degree of freedoms for F statistic

# ANOVA STEP BY STEP

- Step 1: Calculate each treatment mean and the grand mean
- Step 2: Calculate the  $SS_{\text{within}}$ , the  $df_{\text{within}}$ , and the  $s_w^2$
- Step 3: Calculate the  $SS_{\text{between}}$ , the  $df_{\text{between}}$ , and the  $s_b^2$
- Step 4: Calculate F
- Step 5: Compare to critical values of F (using degrees of freedom and the F table)

# ANOVA TABLE

Source	SS	df	$s^2$	F
Between treatments	$SS_b$	$df_b$	$s_b^2 = SS_b/df_b$	$=s_b^2/s_w^2$
Within treatments	$SS_w$	$df_w$	$s_w^2 = SS_w/df_w$	
Total	$SS_{total}$	$df_{total}$		

Source	SS	df	$s^2$	F
Between treatments	32.4	5-1	16.2	12.18
Within treatments	16	3-1	1.33	
Total	48.4	6		

# ANOVA FORMULAS

Source	SS	df	$s^2$	F
Between treatments	$\sum n_i(m_i - m)^2$	$k - 1$	$s_b^2 = SS_b / df_b$	$= s_b^2 / s_w^2$
Within treatments	$\sum (X_{ij} - m_i)^2$	$n - k$	$s_w^2 = SS_w / df_w$	
Total	$\sum (X_{ij} - m)^2$	$n - 1$		

- $X_{ij}$  = an individual observation
- $m_i$  = the mean of  $i^{\text{th}}$  group
- $m$  = the grand mean
- $k$  = the number of groups
- $n_i$  = the number of subjects in the  $i^{\text{th}}$  group
- $n$  = total number of subjects

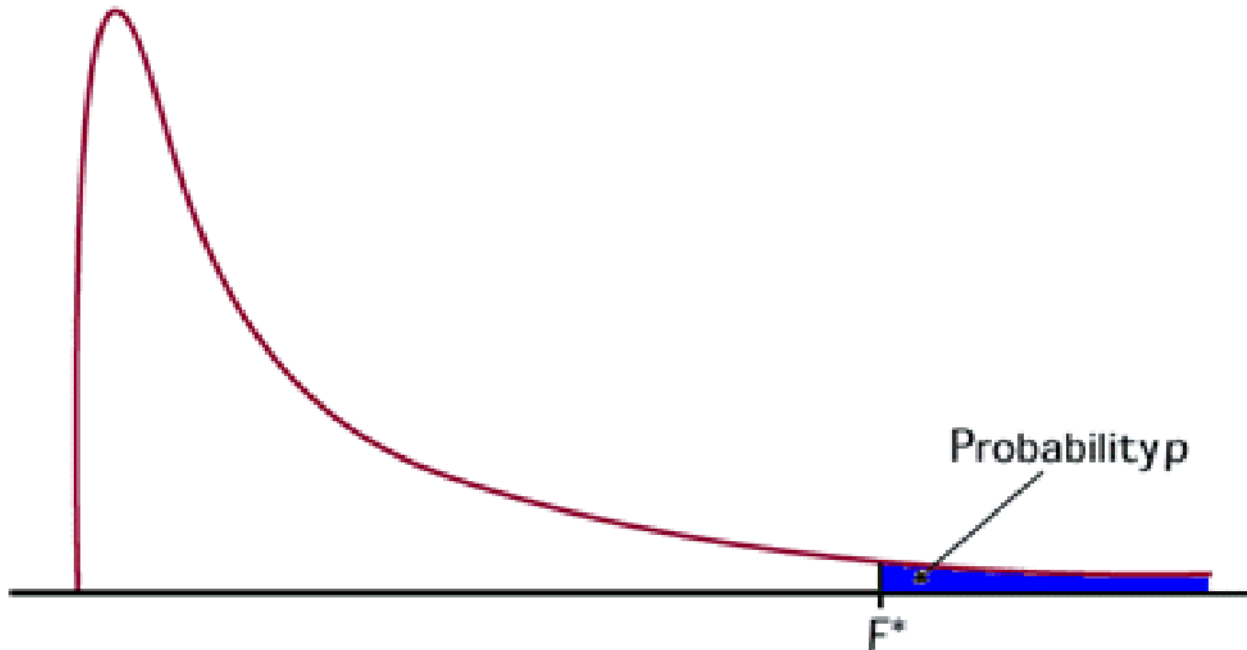
# The F Distribution

- F statistics
  - Because F-ratios are computed from two variances, F values will always be positive numbers.
  - When  $H_0$  is true, the numerator and denominator of the F-ratio are measuring the same variance. In this case the two sample variances should be about the same size, so the ratio should be near 1. In other words, the distribution of F-ratios should pile up around 1.00.

# THE F TABLE

## F Distribution

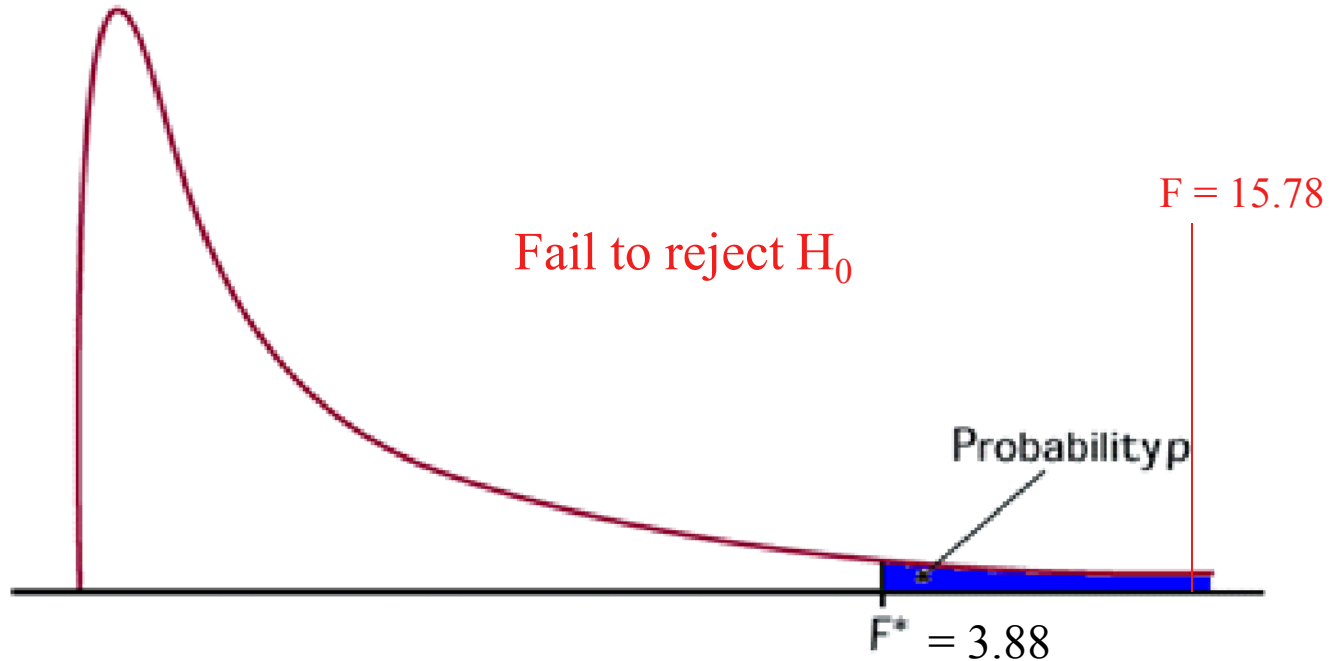
- Positively skewed distribution: Concentration of values near 1, no values smaller than 0 are possible
- Like t, F is a family of curves. Need to use two degrees of freedom (for  $s_b^2$  and  $s_w^2$ ).





# THE F TABLE

For an alpha of 0.05, the critical value of F with degrees of freedom 2.12 is 3.88



# ANOVA ASSUMPTIONS

1. Independent simple random samples
2. The populations are normal
3. The populations have equal variance

# ANOVA BY EXAMPLE

The data depicted below were obtained from an experiment designed to measure the effectiveness of three pain relievers (A, B, and C). A fourth group that received a placebo was also tested.

Placebo	Drug A	Drug B	Drug C
0	0	3	8
0	1	4	5
3	2	5	5

Is there any evidence for a significant difference between groups?

# ANOVA BY EXAMPLE

Placebo	Drug A	Drug B	Drug C
0	0	3	8
0	1	4	5
3	2	5	5

Source	SS	df	MS	F
Between treatments	54	3	18	F = 9
Within treatments	16	8	2	
Total	70	11		

$F_{\text{crit}} = 4.07$   
so fail to reject  $H_0$

# Comparison of Specific Means

The technique to use depends on the nature of the research

## No *a priori* Predictions

If we do not have strong predictions about how the means will differ:

- 1) Run an ANOVA
- 2) If ANOVA is significant, run post-hoc “multiple comparison” tests (e.g. LSD, Bonferroni, Tukey’s)

conservative!

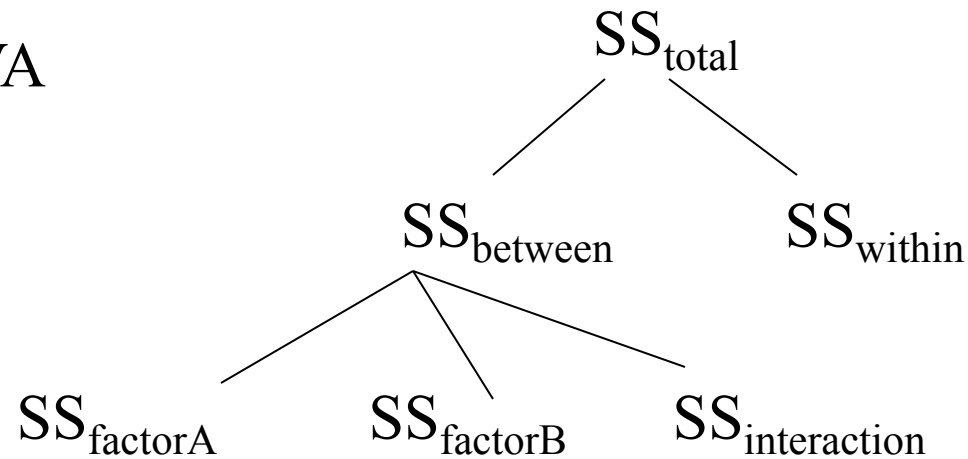
## Yes *a priori* Predictions

If we do have strong predictions about how the means will differ:

- 1) Run contrast tests on the expected differences

# LOGIC OF TWO WAY ANOVA

## Two Way ANOVA



Main Effect 1

$$F = \frac{s_{factorA}^2}{s_w^2}$$

Main Effect 2

$$F = \frac{s_{factorB}^2}{s_w^2}$$

Interaction

$$F = \frac{s_{interaction}^2}{s_w^2}$$

# Logic of Two Way ANOVA

$$SS_{tot} = \sum (X - \bar{\bar{X}})^2$$

$$SS_w = \sum (X - \bar{X}_{cell})^2$$

$$SS_B = \sum n_{cell} (\bar{X}_{cell} - \bar{\bar{X}})^2$$

This is no surprise

$$SS_{rowfactor} = \sum n_{row} (\bar{X}_{row} - \bar{\bar{X}})^2$$

$$SS_{colfactor} = \sum n_{col} (\bar{X}_{col} - \bar{\bar{X}})^2$$

$$SS_{interaction} = SS_{tot} - SS_{within} - SS_{rowf} - SS_{colf}$$

This is news

# STEPS OF TWO WAY ANOVA

	50°	70°	90°	
Morning	1 2 3 $\bar{X} = 2$	4 5 6 $\bar{X} = 5$	3 1 2 $\bar{X} = 2$	$\bar{X} = 3$
Afternoon	1 4 1 $\bar{X} = 2$	0 1 2 $\bar{X} = 1$	0 0 0 $\bar{X} = 0$	$\bar{X} = 1$
	$\bar{X} = 2$	$\bar{X} = 3$	$\bar{X} = 1$	$\bar{\bar{X}} = 2$

Step 1: Get the cell means, row means, column means, and grand mean



# STEPS OF TWO WAY ANOVA

	50°	70°	90°
Morning	1 2 3 $\bar{X} = 2$	4 5 6 $\bar{X} = 5$	3 1 2 $\bar{X} = 2$
Afternoon	1 4 1 $\bar{X} = 2$	0 1 2 $\bar{X} = 1$	0 0 0 $\bar{X} = 0$

$$\bar{\bar{X}} = 2$$

Step 1: Get the  $SS_{\text{total}}$

$$SS_{\text{tot}} = \sum (X - \bar{\bar{X}})^2$$

$$SS_{\text{tot}} = 56$$

# STEPS OF TWO WAY ANOVA

	50°	70°	90°
Morning	1 2 3 $\bar{X} = 2$	4 5 6 $\bar{X} = 5$	3 1 2 $\bar{X} = 2$
Afternoon	1 4 1 $\bar{X} = 2$	0 1 2 $\bar{X} = 1$	0 0 0 $\bar{X} = 0$

$$\bar{\bar{X}} = 2$$

Step 2: Get the  $SS_{\text{within}}$

$$SS_w = \sum (X - \bar{X}_{\text{cell}})^2$$

$$SS_w = 14$$

# STEPS OF TWO WAY ANOVA

	50°	70°	90°	
Morning	1 2 3 $\bar{X} = 2$	4 5 6 $\bar{X} = 5$	3 1 2 $\bar{X} = 2$	$\bar{X} = 3$
Afternoon	1 4 1 $\bar{X} = 2$	0 1 2 $\bar{X} = 1$	0 0 0 $\bar{X} = 0$	$\bar{X} = 1$
				$\bar{\bar{X}} = 2$

Step 3: Get the  $SS_{\text{time}}$

$$SS_{\text{rowfactor}} = \sum n_{\text{row}} (\bar{X}_{\text{row}} - \bar{\bar{X}})^2$$

$$SS_{\text{row-factor}} = 18$$

# STEPS OF TWO WAY ANOVA

	50°	70°	90°	
Morning	1 2 3 $\bar{X} = 2$	4 5 6 $\bar{X} = 5$	3 1 2 $\bar{X} = 2$	
Afternoon	1 4 1 $\bar{X} = 2$	0 1 2 $\bar{X} = 1$	0 0 0 $\bar{X} = 0$	
	$\bar{X} = 2$	$\bar{X} = 3$	$\bar{X} = 1$	$\bar{\bar{X}} = 2$

**Step 4: Get the  $SS_{temp}$**

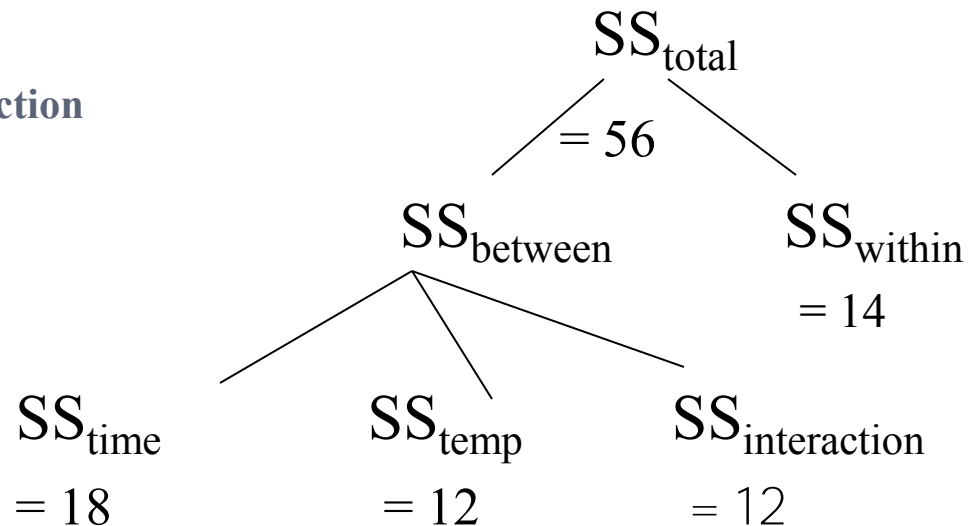
$$SS_{colfactor} = \sum n_{col} (\bar{X}_{col} - \bar{\bar{X}})^2$$

$$SS_{col-factor} = 12$$

# STEPS OF TWO WAY ANOVA

	50°	70°	90°
Morning	1 2 3 $\bar{X} = 2$	4 5 6 $\bar{X} = 5$	3 1 2 $\bar{X} = 2$
Afternoon	1 4 1 $\bar{X} = 2$	0 1 2 $\bar{X} = 1$	0 0 0 $\bar{X} = 0$

Step 5: Get the  $SS_{\text{interaction}}$



# STEPS OF TWO WAY ANOVA

50°

70°

90°

Morning

1 2 3

4 5 6

3 1 2

Afternoon

1 4 1

0 1 2

0 0 0

Morning	1 2 3	4 5 6	3 1 2
Afternoon	1 4 1	0 1 2	0 0 0

## Step 6: Finish the table

Source	SS	df	$s^2$	F	
Between					
$SS_{\text{time}}$	18	1	18	15.38	$p < 0.01$
$SS_{\text{temp}}$	12	2	6	5.12	$p < 0.05$
$SS_{\text{tim} \times \text{temp}}$	12	2	6	5.12	$p < 0.05$
Within	14	12	1.17		
Total	56	17			

# STEPS OF TWO WAY ANOVA

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- Since there are only two levels of time, you know they are significantly different from one another
- Since there are three levels of temp, you will need to use post-hoc multiple comparisons (e.g. Tukey's HSD) to determine which levels differ from one another.
  - To understand the interaction more fully, you might want to look at the "simple effects"
  - You could compute the effect of temp just in the morning, for example.
  - You would simply compute  $SS_{\text{temp at AM}}$

# TWO WAY ANOVA ASSUMPTIONS

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1. Observations within each sample are independent.
2. Populations are normally distributed.
3. Populations from which the samples are selected must have equal variances (homogeneity of variance).
4. Equal cell Ns (if calculating by hand)



<http://www.ncbi.nlm.nih.gov/pubmed/23166130>

*BMJ Open*. 2012 Nov 19;2(6). pii: e001674. doi: 10.1136/bmjopen-2012-001674. Print 2012.

**Life event stress and chronic obstructive pulmonary disease (COPD): associations with mental well-being and quality of life in a population-based study.**

[Lu Y](#), [Nyunt MS](#), [Gwee X](#), [Feng L](#), [Feng L](#), [Kua EH](#), [Kumar R](#), [Ng TP](#).

Gerontological Research Programme, Yong Loo Lin School of Medicine, National University of Singapore, Singapore, Singapore.

**Abstract**

**OBJECTIVES:** To investigate whether life event stress was associated with greater psychological distress and poorer quality of life in older individuals with chronic obstructive pulmonary disease (COPD), in comparison with their counterparts without COPD.

**DESIGN:** Cross-sectional study.

**PARTICIPANTS:** A population-based sample (N=497) of individuals aged 65 and above with COPD (postbronchodilatation FEV1/FVC<0.70, N=136) and without COPD (N=277).

**MEASUREMENTS:** We measured life event stress, depressive symptoms (GDS, Geriatric Depression Scale), cognitive symptoms and function (CFQ, Cognitive Failures Questionnaire and MMSE, Mini-Mental State Examination), and physical and mental health functional status (SF36-PCS, Physical Health Component Summary and SF36-MCS, Mental Health Component Summary) in participants with and without COPD.

**RESULTS:** In two-way analysis of variance controlling for potential confounders, life event stress was associated with significant main effects of worse GDS ( $p<0.001$ ), SF36-PCS ( $p=0.008$ ) and SF36-MCS scores ( $p<0.001$ ), and with significant interaction effects on GDS score ( $p<0.001$ ), SF36-PCS ( $p=0.045$ ) and SF36-MCS ( $p=0.034$ ) in participants with COPD, more than in non-COPD participants. The main effect of COPD was found for postbronchodilator FEV1 ( $p<0.001$ ) and cognitive symptoms ( $p=0.02$ ).

**CONCLUSIONS:** Our findings indicate that life event stress was associated with more depressive symptoms and worse quality of life in individuals with COPD, much more than in those without COPD. Further studies should explore the role of cognitive appraisal of stress, coping resources and psycho-social support in this relationship.

Table 4

	Sum of squares	df	Mean square	Fischer's F value	P value
Analysis of variance for RMNPI at different durations					
Test sites ( <i>n</i> =98)					
Between groups	0.869	2	0.434	289.644	0.000*
Within groups	0.437	291	0.002		
Control sites ( <i>n</i> =91)					
Between groups	0.164	2	0.082	4.129	0.077
Within groups	5.355	270	0.020		
Analysis of variance for interdental bleeding index at different durations					
Test sites ( <i>n</i> =98)					
Between groups	2.313	3	0.771	79.07	0.000*
Within groups	3.783	388	0.010		
Control sites ( <i>n</i> =91)					
Between groups	0.109	3	0.036	1.818	0.144
Within groups	7.209	360	0.020		

\*Significant

Statistical significance using ANOVA for difference between mean values at different durations among the test and control sites

# SUMMARY

## Basics of tests of Variability

- Single Sample
- Two Samples