
PROBABILITIES & RANDOM VARIABLES

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Abraham de Moivre (1667-1754)

“Some of the Problems about Chance having a great appearance of Simplicity, the Mind is easily drawn into a belief, that their Solution may be attained by the meer Strength of natural good Sense; which generally proving otherwise and the Mistakes occasioned thereby being not infrequent, ‘tis presumed that a Book of this Kind, which teaches to distinguish Truth from what seems so nearly to resemble it, will be looked upon as a help to good Reasoning.”

OBJECTIVES

- Basics of probabilities
- Probabilities rules
- Conditional probabilities
- Random variables

BASICS OF PROBABILITIES

- Probability
 - a numerical measure of uncertainty
 - Uncertainty about phenomena or event outcomes could be described in terms such as “unlikely”, “possible”, “likely”, “probable”
 - a way of expressing knowledge or belief that an event will occur or has occurred
 - is synonymous with chance or likelihood
 - Is a numerical value of uncertainty associated with the outcome of an uncertain or unpredictable event

BASICS OF PROBABILITIES

■ Experiment:

- = an activity or investigation for which the results are uncertain
 - Tossing a coin
 - Rolling a die
 - Counting failures over time

■ Outcome:

- = the result of one execution of the experiment

■ Events space = the set of all possible outcomes of an experiment. Symbol: S

BASICS OF PROBABILITIES

- Event spaces:
 - Tossing a coin:
 - Two possible outcomes: head (H) and tails (T)
 - $S = \{H, T\}$
 - Rolling a die:
 - Six possible outcomes: 1, 2, 3, 4, 5, 6
 - $S = \{1, 2, 3, 4, 5, 6\}$
- The event spaces can contain discrete points (such as pass or fail) or continuum points. An event E is a specified set of possible outcomes in a event space S : $E \subset S$, where $\subset =$ “subset of”

BASICS OF PROBABILITIES

- Empty set or null set: a special set without any element. It is denoted by \emptyset .
- Elementary event: the event that could not be decomposed into smaller parts.
- Compound event: the event that could be decomposed into smaller parts.
 - In practical situations most events are compound events

BASICS OF PROBABILITIES

- **Reunion (OR):**

- Symbol: $A \cup B$

- At least one event (A OR B) occurs

- **Intersection (AND):**

- The probability that A and B both occur

- Use the multiplication rule

- Symbol: $A \cap B$

- the events A and B occur simultaneously

- **Negation:**

- Symbol: $\text{non}A$

BASICS OF PROBABILITIES

- Complementary event:
 - The complement of an event E in a event space S is the set of all event points in S that are not in E .
 - Symbol: \bar{E} (read E-bar)
- Mutually exclusive events:
 - Two events E_1 and E_2 in a event space S are said to be mutually exclusive if the event $E_1 \cap E_2$ contains no outcome in the event space S .
 - These two events can not in the same time.

BASICS OF PROBABILITIES

- Independent events:
 - Two events E_1 and E_2 are said to be statistically independent events if the joint probability of E_1 and E_2 equals the product of their marginal probabilities of E_1 and E_2 .
- Union: The union of two events E_1 and E_2 , denoted by $E_1 \cup E_2$, is defined to be the event that contain all sample points in E_1 or E_2 or both.
- Intersection: The intersection of two events E_1 and E_2 , denoted by $E_1 \cap E_2$, is defined to be the event that contain all sample points that are in both E_1 and E_2 .

BASICS OF PROBABILITIES

- Marginal probability: the probability that an event will occur, regardless of whether other events occurs
 - The probability $\Pr(R)$ of a car jumping the red line at a given time at a given intersection, regardless of whether a pedestrian is in the crosswalk
- Conditional probability: of event B, given that event A has occurred, denoted by $\Pr(B|A)$, is defined by:

$$\Pr(B|A) = \Pr(B \cap A) / \Pr(A) \text{ for } \Pr(A) > 0$$

BASICS OF PROBABILITIES

Subjective probability:

- Established subjective (empiric) base on previous experience or on studying large populations
- Implies elementary that are not equipossible (equally likely)

Objective probability:

- Equiprobable outcomes
- Geometric probability

Formula of calculus:

- If an A event could be obtained in S tests out of n equiprobable tests, then the $\Pr(A)$ is given by the number of possible cases
- $\Pr(A) = (\text{no of favorable cases}) / (\text{no of possible cases})$

BASICS OF PROBABILITIES

- A probability of an event A is represented by a real number in the range from 0 to 1 and written as $\Pr(A)$.
- Probabilities are numbers which describe the likelihoods of random events.

$$\Pr(A) \in [0, 1]$$

- Let A be an event:
 - $\Pr(A)$ = the probability of event A
 - If A is certain, then $\Pr(A) = 1$
 - If A is impossible, then $\Pr(A) = 0$

CHANCES AND ODDS

- Chances are probabilities expressed as percent.
 - Range from 0% to 100%.
 - Ex: a probability of 0.65 is the same as a 65% chance.
- The odds for an event is the probability that the event happens, divided by the probability that the event doesn't happen.
 - Can take any positive value
 - Let A be the event. $\text{Odds}(A) = \text{Pr}(A)/[1-\text{Pr}(A)]$, where $1-\text{Pr}(A) = \text{Pr}(\text{non}A)$
 - Example: $\text{Pr}(A) = 0.75$; a probability of 0.75 is the same as 3-to-1 odds ($0.75/(1-0.75)=0.75/0.25=3/1$)

PROPERTIES OF PROBABILITIES

- Take values between 0 and 1:

$$0 \leq \Pr(A) \leq 1$$

- $\Pr(\text{event space}) = 1$

- The probability that something happens is one minus the probability that it does not:

$$\Pr(A) = 1 - \Pr(\text{non}A)$$

PROBABILITY RULES

- **Addition Rule**: probability of A **or** B:

$$\Pr(\mathbf{A \text{ or } B}) = \Pr(\mathbf{A}) + \Pr(\mathbf{B})$$

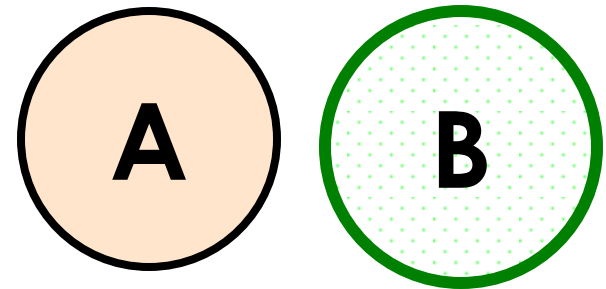
when A and B are mutually exclusive

- **Multiplication Rule**: probability of A **and** B:

$$\mathbf{P(A \text{ and } B)} = \mathbf{P(A)} \cdot \mathbf{P(B)}$$

when A and B are independent

PROBABILITY RULES: ADDITION RULE



- Let A and B be two events:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

$$\Pr(\text{A or B}) = \Pr(A) + \Pr(B) - \Pr(\text{A and B})$$

- A and B mutually exclusive:

- $\Pr(A \cap B) = 0$

- $\Pr(\text{A and B}) = 0$

PROBABILITY RULES: ADDITION RULE

- $A = \{\text{SBP of mother} > 140 \text{ mmHg}\}$
 - $\Pr(A) = 0.25$
- $B = \{\text{SBP of father} > 140 \text{ mmHg}\}$
 - $\Pr(B) = 0.15$
- What is the probability that mother or father to have hypertension?

$$\Pr(A \cup B) = 0.25 + 0.15 - 0 = 0.40$$

$$\Pr(A \text{ or } B) = 0.25 + 0.15 - 0 = 0.40$$

PROBABILITY RULES: ADDITION RULE

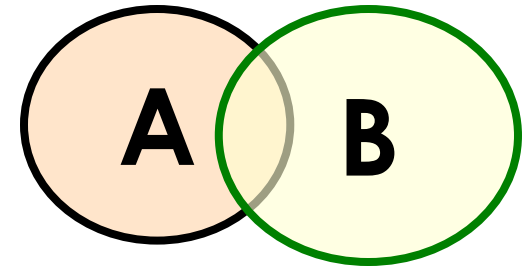
- In a cafe are at a moment 20 people, 10 like tea, 10 like coffee and other 2 like tea and coffee.
- What is probability to random extract from this population one person who like tea or coffee?

$$\Pr(\text{tea} \cup \text{coffee}) = \Pr(\text{tea}) + \Pr(\text{coffee}) - \Pr(\text{tea} \cap \text{coffee})$$

$$\Pr(\text{tea or coffee}) = \Pr(\text{tea}) + \Pr(\text{coffee}) - \Pr(\text{tea and coffee})$$

$$\Pr(\text{tea or coffee}) = 0.50 + 0.50 - 0.10 = 0.90$$

PROBABILITY RULES: MULTIPLICATION RULE



- Let A and B be two events:

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B|A)$$

$$\Pr(A \text{ and } B) = \Pr(A) \cdot \Pr(B|A)$$

- Independent events: $\Pr(B|A) = \Pr(B)$

PROBABILITY RULES: MULTIPLICATION RULE

- $A = \{\text{SBP of mother} > 140 \text{ mmHg}\}$
 - $\Pr(A) = 0.10$
- $B = \{\text{SBP of father} > 140 \text{ mmHg}\}$
 - $\Pr(B) = 0.20$
- $\Pr(A \cap B) = 0.05$; $\Pr(A \text{ and } B) = 0.05$
- The two events are dependent or independent?

$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$ - independent events

$0.05 \neq 0.10 \cdot 0.20 \rightarrow$ the events are dependent

BAYES' THEOREM

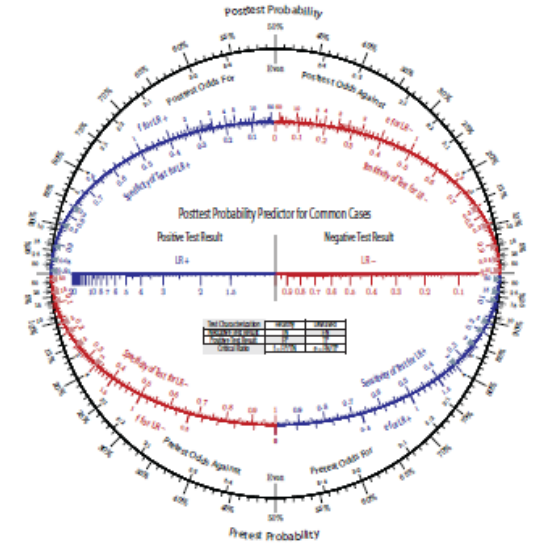
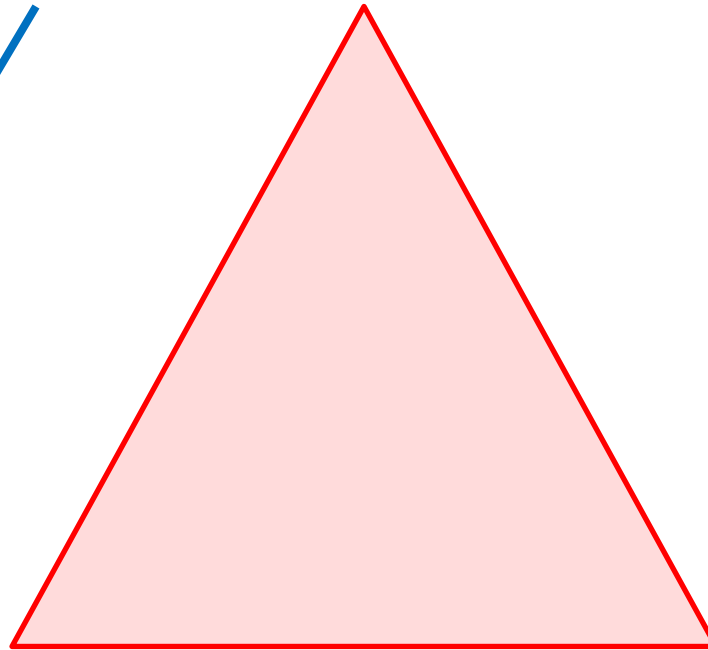
- Bayes' law or Bayes' rule
- Gives the relationship between the probabilities of event A and event B and the conditional probabilities of A given B and B given A:

- $$\Pr(A|B) = \frac{\Pr(B|A) \times \Pr(A)}{\Pr(B|A) \times \Pr(A) + \Pr(B|nonA) \times \Pr(nonA)}$$

BAYES' THEOREM

Diagnosis

It's all about probabilities



Bayer's rule



Nomography (Modern)

Math for updating probabilities

Graphical technique for doing the math

$$Pr_{old} + Test = Pr_{new}$$

CONDITIONAL PROBABILITIES

- Conditional probability: example
 - (Tuberculin Test+|TBC) is the probability of obtaining a positive tuberculin test to a patient with tuberculosis
- **Pr(B|A)** is not the same things as **Pr(A|B)**

CONDITIONAL PROBABILITIES

- Let:
 - $A = \{\text{TBC}+\}$
 - $B = \{\text{Tuberculin Test}+\}$

	TBC+	TBC-
Test+	15	12
Test-	25	18

- $\Pr(A) = (15+25)/(15+12+25+18) = 0.57$ (prevalence)
- $\Pr(\text{non}A) = (12+18)/(15+12+25+18) = 0.43$
- $\Pr(B|A) =$ probability of a positive tuberculin test to a patient with TBC $= 15/(15+25) = 0.38 =$ **Sensibility (Se)**

CONDITIONAL PROBABILITIES

- Let:
 - $A = \{\text{TBC}+\}$
 - $B = \{\text{Tuberculin Test}+\}$

	TBC+	TBC-
Test+	15	12
Test-	25	18

- $\Pr(\text{non}B|\text{non}A)$ = probability of obtaining a negative test to a patient without TBC = $18/(18+12) = 0.60 = \mathbf{\text{Specificity (Sp)}}$
- $\Pr(A|B)$ = probability that a person with TBC to have a positive tuberculin test = $15/(15+12) = 0.56 = \mathbf{\text{Predictive Positive Value (PPV)}}$

CONDITIONAL PROBABILITIES

- Let:
 - A = {TBC+}
 - B = {Tuberculin Test+}

	TBC+	TBC-
Test+	15	12
Test-	25	18

- $\Pr(\text{nonA}|\text{nonB})$ = probability that a person without TBC to have a negative tuberculin test = $18/(18+25) = 0.42 =$
Negative Predictive Value (NPV)
- Positive False Ratio: $\text{PFR} = \Pr(B|\text{nonA})$
- Negative False Ratio: $\text{RFN} = \Pr(\text{nonA}|B)$

INDEPENDENT EVENTS: CONDITIONAL PROBABILITIES

- Events A and B are independent if the probability of event B is the same whether or not A has occurred.

- Two events A and B are Independent IF

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$$

- If (and only if) A and B are independent, then:

- $\Pr(B|A) = \Pr(B|\text{non}A) = \Pr(B)$

- $\Pr(A|B) = \Pr(A|\text{non}B) = \Pr(A)$

- It expressed the independence of the two events: the probability of event B (respectively A) did not depend by the realization of event A (respectively B)

- Example: if a coin is toss twice the probability to obtain “head” to the second toss is always 0.5 and is not depending if at the first toss we obtained “head” or “tail”.

RANDOM VARIABLE

- A random variable
 - is a quantification of a probability model that allows to model random data
 - Is a quantity that may take any value of a given range that cannot be predicted exactly but can be described in terms of their probability
 - Discrete: generally assessed by counting
 - Continuous: generally assessed by measurements

RANDOM VARIABLE

- When throwing a die with 6 faces, let X be the random variable defined by:

X = the square of the scores shown on the die

What is the expectation of X ?

Solution:

- $S = \{1, 2^2, 3^2, 4^2, 5^2, 6^2\} = \{1, 4, 9, 16, 25, 36\}$
- Each face has a probability of $1/6$ of occurring, so:

$$E(X) = 1 \cdot 1/6 + 4 \cdot 1/6 + 9 \cdot 1/6 + 16 \cdot 1/6 + 25 \cdot 1/6 + 36 \cdot 1/6 = 91 \cdot 1/6$$

REMEMBER!

■ Addition rules:

- $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
- $\Pr(A \text{ or } B) = \Pr(A) + \Pr(B) - \Pr(A \text{ and } B)$
- Mutually exclusive events:
 - $\Pr(A \cup B) = \Pr(A) + \Pr(B)$
 - $\Pr(A \text{ or } B) = \Pr(A) + \Pr(B)$:

■ Multiplication Rule:

- $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B|A)$
- $\Pr(A \text{ and } B) = \Pr(A) \cdot \Pr(B|A)$
- Independent events:
 - $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$