PROBABILITIES
CONDITIONAL PROBABILITIES

Sorana D. Bolboacă

Abraham de Moivre (1667-1754)

"Some of the Problems about Chance having a great appearance of Simplicity, the Mind is easily drawn into a belief, that their Solution may be attained by the meer Strength of natural good Sense; which generally proving otherwise and the Mistakes occasioned thereby being not infrequent, 'tis presumed that a Book of this Kind, which teaches to distinguish Truth from what seems so nearly to resemble it, will be looked upon as a help to good Reasoning."

OBJECTIVES

- Basics of probabilities. Events space.
- Probabilities rules
- Conditional probabilities. Risks and rates

BASICS OF PROBABILITIES

- Probability
  - a numerical measure of uncertainty
    - Uncertainty about phenomena or event outcomes could be described in terms such as “unlikely”, “possible”, “likely”, “probable”
  - a way of expressing knowledge or belief that an event will occur or has occurred
  - is synonymous with chance or likelihood
  - Is a numerical value of uncertainty associated with the outcome of an uncertain or unpredictable event

BASICS OF PROBABILITIES

- Experiment:
  - = an activity or investigation for which the results are uncertain
    - Tossing a coin
    - Rolling a die
    - Counting failures over time
- Outcome:
  - = the result of one execution of the experiment
- Events space = the set of all possible outcomes of an experiment.
  - Symbol: S

BASICS OF PROBABILITIES

- Event spaces:
  - Tossing a coin:
    - Two possible outcomes: head (H) and tails (T)
    - S = {H, T}
  - Rolling a die:
    - Six possible outcomes: 1, 2, 3, 4, 5, 6
    - S = {1, 2, 3, 4, 5, 6}
- The event spaces can contain discrete points (such as pass or fail) or continuum points. An event E is a specified set of possible outcomes in a event space S: $E \subset S$, where $\subset$ = “subset of”

BASICS OF PROBABILITIES

- Empty set of null set: a special set without any element. It is denoted by $\emptyset$.
- Elementary event: the event that could not be decomposed into smaller parts.
- Compound event: the event that could be decomposed into smaller parts.
  - In practical situations most events are compound events

Basics of Probabilities

- **Reunion (OR):**
  - Symbol: $A \cup B$
    - At least one event (A OR B) occurs
- **Intersection (AND)**:
  - The probability that A and B both occur
  - Use the multiplication rule
  - Symbol: $A \cap B$
    - the events A and B occur simultaneously
- **Negation:**
  - Symbol: $\bar{A} = \text{nonA}$

BASIC OF PROBABILITIES

- Complementary event:
  - A complement of an event E in a event space S is the set of all event points in S that are not in E.
  - Symbol: $\overline{E}$ (read E-bar)
- Mutually exclusive events:
  - Two events E_1 and E_2 in a event space S are said to be mutually exclusive if the event $E_1 \cap E_2$ contain no outcome in the event space S.
  - These two events can not happen in the same time.

BASICS OF PROBABILITIES

- Independent events:
  - Two events E_1 and E_2 are said to be statistically independent events if the joint probability of E_1 and E_2 equals the product of their marginal probabilities of E_1 and E_2.
- Union: The union of two events E_1 and E_2, denoted by $E_1 \cup E_2$, is defined to be the event that contain all sample points in E_1 or E_2 or both.
- Intersection: The intersection of two events E_1 and E_2, denoted by $E_1 \cap E_2$, is defined to be the event that contain all sample points that are in both E_1 and E_2.

BASICS OF PROBABILITIES

- Marginal probability: the probability that an event will occur, regardless of whether other events occurs
  - The probability Pr(R) of a car jumping the red line at a given time at a given intersection, regardless of whether a pedestrian is in the crosswalk
- Conditional probability: of event B, given that event A has occurred, denoted by Pr(B|A), is defined by:

$$Pr(B|A) = Pr(B \cap A)/Pr(B) \text{ for } Pr(B)>0$$

BASICS OF PROBABILITIES

Subjective probability:

- Established subjective (empiric) base on previous experience or on studying large populations
- Implies elementary that are not equipossible (equally likely)

Objective probability:

- Equiprobable outcomes
- Geometric probability

Formula of calculus:

- If an A event could be obtained in S tests out of n equiprobable tests, then the Pr(A) is given by the number of possible cases
- Pr(A) = (no of favorable cases)/(no of possible cases)

BASICS OF PROBABILITIES

- A probability of an event A is represented by a real number in the range from 0 to 1 and written as Pr(A).
- Probabilities are numbers which describe the likelihoods of random events.

$$\mathbf{Pr(A) \in [0, 1]}$$

- Let A be an event:
  - Pr(A) = the probability of event A
  - If A is certain, then Pr(A) = 1
  - If A is impossible, then Pr(A) = 0

CHANCES AND ODDS

- Chances are probabilities expressed as percent.
  - Range from 0% to 100%.
  - Ex: a probability of 0.65 is the same as a 65% chance.

- The odds for an event is the probability that the event happens, divided by the probability that the event doesn't happen.
  - Can take any positive value
  - Let A be the event. Odds(A) = Pr(A)/[1-Pr(A)], where 1-Pr(A) = Pr(nonA)
  - Example: Pr(A) = 0.75; a probability of 0.75 is the same as 3-to-1 odds (0.75/(1-0.75)=0.75/0.25=3/1)

PROPERTIES OF PROBABILITIES

- Take values between 0 and 1:

$$0 \le Pr(A) \le 1$$

- Pr(event space) = 1
- The probability that something happens is one minus the probability that it does not:

$$Pr(A) = 1 - Pr(\bar{A})$$

- **Addition Rule**: probability of A **or** B:

$$Pr(A \cup B) = Pr(A) + Pr(B)$$

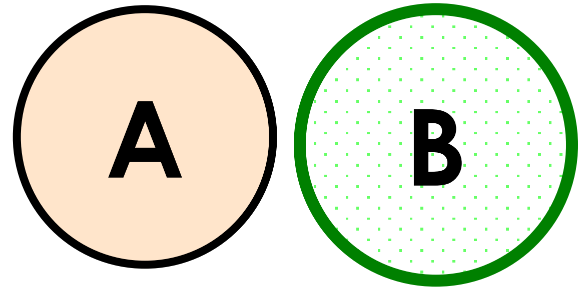
when A and B are mutually exclusive

- **Multiplication Rule**: probability of A **and** B:

$$P(A \cap B) = P(A) \cdot P(B)$$

when A and B are independent

Probability Rules: Addition Rule



- Let A and B be two events:

$$\mathbf{Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)}$$

$$\mathbf{Pr(A\ or\ B) = Pr(A) + Pr(B) - Pr(A\ and\ B)}$$

- A and B mutually exclusive:
  - **$Pr(A \cap B) = 0$**
  - **Pr(A and B) = 0**
- A = {SBP of mother > 140 mmHg}
  - Pr(A) = 0.25
- B = {SBP of father > 140 mmHg}
  - Pr(B) = 0.15
- What is the probability that mother or father to have hypertension?

$$\mathbf{Pr(A \cup B) = 0.25 + 0.15 - 0 = 0.40}$$

$$\mathbf{Pr(A\ or\ B) = 0.25 + 0.15 - 0 = 0.40}$$

PROBABILITY RULES: ADDITION RULE

- In a cafe are at a moment 20 people, 10 like tea, 10 like coffee and 2 like tea and coffee.
- What is probability to random extract from this population one person who like tea or coffee?

$$\mathbf{Pr(tea \cup coffee) = Pr(tea) + Pr(coffee) - Pr(tea \cap coffee)}$$

$$\mathbf{Pr(tea\ or\ coffee) = Pr(tea) + Pr(coffee) - Pr(tea\ and\ coffee)}$$

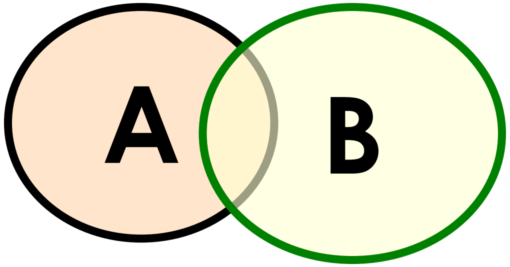
$$\mathbf{Pr(tea \cup coffee) = 0.50 + 0.50 – 0.10 = 0.90}$$

PROBABILITY RULES: MULTIPLICATION RULE

- Let A and B be two events:

$$\mathbf{Pr(A \cap B) = Pr(A) \cdot Pr(B|A)}$$

$$\mathbf{Pr(A\ and\ B) = Pr(A) \cdot Pr(B|A)}$$



- Independent events: **Pr(B|A) = Pr(B)**

- A = {SBP of mother > 140 mmHg}
  - Pr(A) = 0.10
- B = {SBP of father > 140 mmHg}
  - Pr(B) = 0.20
- $Pr(A \cap B) = 0.05$; Pr(A and B) = 0.05
- The two events are dependent or independent?

$$\mathbf{Pr(A \cap B) = Pr(A) \cdot Pr(B)}\ \text{– independent events}$$

$$0.05 \neq 0.10*0.20 \rightarrow \text{the events are dependent}$$

Bayes' Theorem

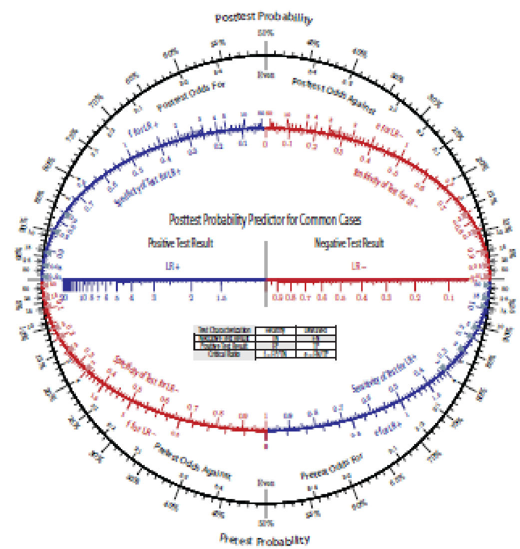
- Bayes' law / Bayes' rule
- Gives the relationship between the probabilities if event A and event B and the conditional probabilities of A given B and B given A:

$$\Pr(A|B) = \frac{\Pr(B|A) \times \Pr(A)}{\Pr(B|A) \times \Pr(A) + \Pr(B|nonA) \times \Pr(nonA)}$$

BAYES' THEOREM

Diagnosis

It's all about probabilities



Bayer's rule → Nomography (Modern)

Math for updating probabilities

Pr_{old} + Test = Pr_{new}

Graphical technique for doing the math

Bayesian Theorem

- Tests are not events
- Tests detect things that do not exists (false positive) and could miss thinks that exists (false negative)
- Tests give us test probabilities NOT the real probability
- False positive skew results
- People prefer natural numbers as 100 in 10.000 rather than 1%
- Even science is a test

Bayesian Theorem

- Bayes' theorem finds the actual probability of an event from the results of a test
  - Correct the measurement errors
    - If you know the real probability & the chance of false positive and false negative
  - Relate the actual probability to the measured test probability.
    - Given mammogram test results and known error rates, you can predict the actual chance of having cancer

Anatomy of a test

- 2% of women have breast cancer
- 70% of mammograms detect breast cancer when it is present → 30% of mammograms miss the diagnosis
- 10% of mammograms incorrectly detect breast cancer when it is **not** present → 90% of mammograms correctly return a negative result

	Breast cancer + (2%)	Breast cancer - (98%)
Mammo +	70	10
Mammo-	30	90

Anatomy of a test: how to read it!

- 2% of women have breast cancer
- If you already have breast cancer, there is 70% chance that your mammogram will be positive and 30% chance that your test will to be negative
- If you do not have breast cancer, there is 10% chance that your mammogram will be positive and 90% chance that your test will to be negative

	Breast cancer + (2%)	Breast cancer - (98%)
Mammo +	70	10
Mammo-	30	90

Anatomy of a test

- Suppose that you have a patient with a positive result. What are her chances to have breast cancer?

	Breast cancer + (2%)	Breast cancer - (98%)
Mammo +	70	10
Mammo-	30	90

- The probability of a **true positive** = the probability to have breast cancer × probability that the mammogram to be positive = 0.02 × 0.7 = 0.014 → 1.4% chance
- The probability of a **false positive** = the probability not to have breast cancer × probability that the mammogram to be positive = 0. 98 × 0.10 = 0.098 → 9.8% chance

Anatomy of a test

	Breast cancer + (2%)	Breast cancer - (98%)
Mammo +	True positive 0.02*0.70 = 0.014	False positive 0.10*0.98=0.098
Mammo-	False negative 0.02*30 = 0.600	True negative 0.90*0.98 = 0.882

- What is the chance that your patient to really have cancer if she get a positive mammogram.
  - The chance of an event is the number of ways it could happen given all possible outcomes:
  - Probability = (desired event)/ (all possibilities)
  - Probability = 0.014/(0.014+0.098) = 0.125 → the chance that your patient to have breast cancer if the mammogram is positive is 12.5%

Anatomy of a test

- → a positive mammogram only means that the individual chance of breast cancer is 12.5%, rather than expected 70%
- → the mammogram gives a false positive 10% of the time → there will be false positives in any given population
- → the problem can be turned into an equation = Bayes' Theorem

Bayes' Theorem

$$\Pr(A \mid B) = \frac{\Pr(B \mid A) \times P(A)}{\Pr(B \mid A) \times \Pr(A) + \Pr(B \mid nonA) \times \Pr(nonA)}$$

$$\Pr(A \mid B) = \frac{\Pr(B \mid A) \times P(A)}{\Pr(B)}$$

- P(A|B) = probability of having breast cancer (A) given a positive mammogram (B)
- P(B|A) = probability of a positive mammogram (B) given that breast cancer is present (A)
- P(A) = probability of having breast cancer → P(nonA) = probability not to have cancer
- P(B|nonA) = probability of a positive mammogram

Prior vs posterior probability

- Prior probability
  - what you believe before seeing any data
- Posterior probability
  - P(hypothesis|data) = the probability of a hypothesis given the data
  - depends on prior probability & observed data

Bayesian inference

- Prior probability + data → posterior probability

Published scientific literature / National statistical data

- Pr(hypothesis is true | observed data)

A good prior is helps, a bad prior hurts, but the prior matter less the more data you have!

CONDITIONAL PROBABILITIES

- Conditional probability: example
  - (Tuberculin Test+|TBC) is the probability of obtaining a positive tuberculin test to a patient with tuberculosis
- Pr(B|A) is not the same things as Pr(A|B)

	TBC+	TBC-
Test+	15	12
Test-	25	18

- Let:
  - A = {TBC+}
  - B = {Tuberculin Test+}
- Pr(A) = (15+25)/(15+12+25+18) = 0.57 (prevalence)
- $\Pr(\bar{A})$ = (12+18)/(15+12+25+18) = 0.43
- Pr(B|A) = probability of a positive tuberculin test to a patient with TBC = 15/(15+25) = 0.38 = **Sensibility (Se)**

CONDITIONAL PROBABILITIES

- Let:
  - A = {TBC+}
  - B = {Tuberculin Test+}

	TBC+	TBC-
Test+	15	12
Test-	25	18

- Pr(nonB|nonA) = probability of obtaining a negative test to a patient without TBC = 18/(18+12) = 0.60 = **Specificity (Sp)**
- Pr(A|B) = probability that a person with TBC to have a positive tuberculin test = 15/(15+12) = 0.56 = Predictive Positive Value **(PPV)**

Conditional Probabilities

- Let:
  - A = {TBC+}
  - B = {Tuberculin Test+}

	TBC+	TBC-
Test+	15	12
Test-	25	18

- Pr(nonA|nonB) = probability that a person without TBC to have a negative tuberculin test = 18/(18+25) = 0.42 = **Negative Predictive Value (NPV)**
- Positive False Ratio: PFR = $\Pr(B|\bar{A})$
- Negative False Ratio: RFN = $\Pr(\bar{A}|B)$

INDEPENDENT EVENTS: CONDITIONAL PROBABILITIES

- Events A and B are independent if the probability of event B is the same whether or not A has occurred.
- Two events A and B are Independent IF

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$$

- If (and only if) A and B are independent, then:
  - $\Pr(B|A) = \Pr(B|\bar{A}) = \Pr(B)$
  - $\Pr(A|B) = \Pr(A|\bar{B}) = \Pr(A)$
- It expressed the independence of the two events: the probability of event B (respectively A) did not depend by the realization of event A (respectively B)
  - Example: if a coin is toss twice the probability to obtain "head" to the second toss is always 0.5 and is not depending if at the first toss we obtained "head" or "tail".

RECALL!

- **Addition rules:**
  - $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$

    Pr(A or B) = Pr(A) + Pr(B) – Pr(A and B)
  - Mutually exclusive events:
    - $Pr(A \cup B) = Pr(A) + Pr(B)$
    - Pr(A or B) = Pr(A) + Pr(B):
- **Multiplication Rule:**
  - $Pr(A \cap B) = Pr(A) \cdot Pr(B|A)$

    Pr(A and B) = Pr(A)·Pr(B|A)
  - Independent events:
    - $Pr(A \cap B) = Pr(A) \cdot Pr(B)$

Thank you!